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On iteration groups of C^1-diffeomorphisms with two fixed points

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ABSTRACT

Let $I=[0,1]$ and let $f: I \to I$ satisfy the following general hypothesis: $f \in Diff^1[I]$, $f(0)=0$, $f(1)=1$, $f(x) \neq x$, $x \in Int\, I$ and the condition (A): $f'(x)=s+0(x^\delta)$, $x \to 0$, and $f'(x)=M+0(|x-1|^\delta)$, $x \to 1$, with $s<1<M$. If $\{f^t: I \to I,\ t \in \mathbb{R}\}$ is a continuous iteration group where all functions are of class C^1 and at least one of the iterates satisfies condition (A), then there exists a diffeomorphism $\psi: I \to I$ such that $f^t=\psi^{-1} \circ p_k^t \circ \psi$, $t \in \mathbb{R}$, where

$$p_k^t(x) := \frac{s^t x}{[1+(s^{tk}-1)x^k]^{1/k}}, \quad x \in I \text{ and } k = \frac{-\log M}{\log s}.$$

The function ψ is given by the formula

$$\psi(x) = \lim_{n\to\infty} \frac{f^n(x)}{[(f^n(x_0))^k + (f^n(x))^k]^{1/k}}, \quad x \in I \text{ with an arbitrary fixed } x_0 \in Int\, I.$$

Giving an example of a C^1-iteration group with two fixed points which does not conjugate diffeomorphically with the group $\{p_k^t: I \to I,\ t \in \mathbb{R}\}$ we show that some additional assumption on diffeomorphism f is essential.

1. Introduction

Let I be an interval and $f: I \to I$ be a homeomorphism of class C^r, where $r \geqslant 0$. A family of homeomorphisms of class C^r $\{f^t: I \to I,\ t \in \mathbb{R}\}$ such that $f^t(f^s(x)) = f^{t+s}(x)$, $t, s \in \mathbb{R}$, $f^1(x) = f(x)$, $x \in I$, and the mappings $t \to f^t(x)$ are continuous for $x \in I$ is said to be *a C^r-iteration group of the function f* or *C^r-flow of f*. The problem of the embeddability of a given function with two fixed points in a regular iteration group was considered in branching processes for probability generating functions in [4] and meromorphic functions in [5] by Karlin and McGregor. For the case when $f \in Diff^1[I]$ possesses the only one fixed point $p \in I$, $0 < f'(p) =: s < 1$ and $f'(x) = s + O(|x-p|^\delta)$, $x \to p$, $\delta > 0$ it is known that then there exists the exactly one C^1-iteration group of f. This group is linearizable, i.e. diffeomorphically conjugated with the group $\{s^t id,\ t \in \mathbb{R}\}$ of linear mappings which we will call its canonical group. This means that there exists a diffeomorphism $\varphi: I \to \mathbb{R}$ such that

$$f^t = \varphi^{-1} \circ s^t id \circ \varphi, \quad t \in \mathbb{R}$$

(see [7,10]). The aim of the present paper is to examine a similar problem for C^1-iteration groups of a function f with two fixed points. We will show that these groups are not diffeomorphically linearizable. We consider a special family of two

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parameters nonlinear iteration groups built of possible simplest elementary functions with two fixed points, such that every sufficiently regular iteration group with two fixed points is diffeomorphically conjugated with one of these special groups. We will call this simple group the canonical group of a given flow.

It seems to be natural to start such considerations with the group of homographies with two fixed points 0 and 1, namely with $\{q^t,\ t \in \mathbb{R}\}$, where

$$q^t(x) := \frac{s^t x}{1 + (s^t - 1)x}, \quad x \in [0,1],\ t \in \mathbb{R}.$$

But there are some disadvantages in such a case – in that family for every function f of it we have $f'(1) = \frac{1}{f'(0)}$, that means values of the derivatives at fixed points are not able to be arbitrary. To improve this phenomena we construct a family of iteration groups $\{p_k^t,\ t \in \mathbb{R}\}$, where $k > 0$ which are conjugated with $\{q^{kt},\ t \in \mathbb{R}\}$ by the power function x^k. This family is given by the formula

$$p_k^t(x) := \frac{s^t x}{(1 + (s^{tk} - 1)x^k)^{\frac{1}{k}}}, \quad x \in [0,1],\ t \in \mathbb{R}. \tag{1}$$

Now $f'(1) = \frac{1}{(f'(0))^k}$ for every function from this family, so for a given $0 < s < 1$ and $M > 1$ we can choose suitable a $k > 0$ such that $(p_k^t)'(0) = s^t$ and $(p_k^t)'(1) = M^t$, $t \in \mathbb{R}$.

We consider the following problem: When a C^1-iteration group of a function with two fixed points has to be diffeomorphically conjugated to a suitable canonical group $\{p_k^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$? A partial answer of this problem one can find in [2, Theorem 2.19] where was proved that for $r \geqslant 2$ all C^r-flows with exactly two hyperbolic fixed points are C^r-conjugated. In this paper we consider the case $r = 1$. We show that in this case such a result need not to hold.

Let us accept the following general hypotheses:

(H) $f : [0,1] \to [0,1]$ is continuous, strictly increasing, $f(0) = 0,\ f(1) = 1$ and $f(x) \neq x$, $x \in (0,1)$

and

(R) $f \in \mathit{Diff}^1[0,1]$ satisfies (H) and for a $\delta > 0$, $f'(x) = s + O(x^\delta)$, $x \to 0$, $f'(x) = M + O((x-1)^\delta)$, $x \to 1$, $s < 1 < M$.

The answer to the set problem is given in the following theorems:

Theorem 1. *Let f satisfy* (R)*. If $\{f^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$ is a C^1-iteration group of f, then*

$$f^t = \psi^{-1} \circ p_k^t \circ \psi, \quad t \in \mathbb{R}, \tag{2}$$

where $k = -\frac{\log M}{\log s}$ and $\psi : [0,1] \to [0,1]$ is a diffeomorphism given by

$$\psi(x) = \lim_{n\to\infty} \frac{f^n(x)}{[(f^n(x_0))^k + (f^n(x))^k]^{1/k}}, \quad x \in [0,1] \tag{3}$$

as well as

$$\psi(x) = \lim_{n\to\infty} \left[\frac{f^{-n}(x_0) - 1}{f^{-n}(x_0) + f^{-n}(x) - 2}\right]^{1/k}, \quad x \in [0,1], \tag{4}$$

with an arbitrarily fixed $x_0 \in (0,1)$.

Theorem 2. *There exist C^1-iteration groups of a function f satisfying* (H) *such that $(f^t)'(0) \neq 1$, $(f^t)'(1) \neq 1$ for $t \neq 0$ which are not diffeomorphically conjugated to any group $\{p_k^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$, $k > 0$.*

2. Preliminaries

We start with some remarks and lemmas.

Remark 3. Let $f, g : [0,1] \to [0,1]$ be topologically conjugated, that is there exists a homeomorphism $\varphi : [0,1] \to [0,1]$ such that $\varphi \circ f = g \circ \varphi$. If f has only two fixed points 0 and 1, then g also has two fixed points 0 and 1. If, additionally, f, g, φ are differentiable, $\varphi'(0) > 0$ and $\varphi'(1) > 0$, then

$$f'(0) = g'(0) \quad \text{and} \quad f'(1) = g'(1).$$

Note that this simple remark implies that iteration groups $\{p_k^t,\ t \in \mathbb{R}\}$ and $\{s^t id,\ t \in \mathbb{R}\}$ are not topologically conjugate, so the first one is not topologically linearizable.

Remark 4. Let G be differentiable at 0 and 1, $G(0) = 0$, $G(1) = 1$ and

$$\big(G(x)\big)^k = H\big(x^k\big), \quad x \in [0,1] \tag{5}$$

for a $k > 0$. Then $G'(1) = H'(1)$ and $(G'(0))^k = H'(0)$.

Proof. Differentiating of both sides of (5) at $x = 1$ gives the first of the equalities of the thesis. For the second one of them rewrite (5) in the form

$$\left(\frac{G(x) - G(0)}{x - 0}\right)^k = \frac{H(x^k) - H(0)}{x^k - 0}.$$

Letting $x \to 0$ gives the desirable formula. □

Passing to the considerations on iteration groups let us note the obvious

Remark 5. Let

$$q^t(x) := \frac{s^t x}{1 + (s^t - 1)x}, \quad x \in [0,1],\ t \in \mathbb{R}.$$

$\{q^t,\ t \in \mathbb{R}\}$ is a continuous iteration group such that

$$\big(q^t\big)'(0) = s^t \quad \text{and} \quad \big(q^t\big)'(1) = \frac{1}{s^t}.$$

Lemma 6. *Let $s \neq 0, 1$ and $k > 0$. The family $\{p_k^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$, where p_k^t are defined by (1) is a continuous iteration group such that $p_k^t(0) = 0$, $p_k^t(1) = 1$ and $(p_k^t)'(0) = s^t$, $(p_k^t)'(1) = s^{-kt}$, $t \in \mathbb{R}$.*

Proof. The first two equalities are obvious. To prove two last of them it is enough to see that

$$p_k^t = w_k^{-1} \circ q^{kt} \circ w_k, \quad t \in \mathbb{R},$$

where $w_k(x) := x^k$, and apply Remarks 3 and 4. □

Lemma 7. *Let $S^t : [0,\infty] \to [0,\infty]$, $t \in \mathbb{R}$, and $H_k : [0,1] \to [0,\infty]$, $k > 0$, be given by formulas $S^t(x) := s^t x$, $x \in [0,\infty)$, $S^t(\infty) := \infty$ and $H_k(x) := \frac{x}{(1-x^k)^{\frac{1}{k}}}$, $x \in [0,1)$, $H_k(1) := \infty$. Then*

$$S^t = H_k \circ p_k^t \circ H_k^{-1}, \quad t \in \mathbb{R}.$$

Proof. It is easy to see that $H_k^{-1}(x) = \frac{x}{(1+x^k)^{\frac{1}{k}}}$. The proof needs only simple calculations. □

Remark 8. $H_k = w_k^{-1} \circ H \circ w_k$, where $H(x) = \frac{x}{1-x}$.

3. C^0-iteration groups

To prove the main results we need to use some properties of continuous iterations groups of a function with one and two fixed points. Let us start with a quotation of a basic result on the form of iteration groups.

Lemma 9. *(See [1, p. 248], [6, p. 198], [2, p. 68], [11].) Let $0 < s < 1 < M$. If $\{f^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$ is a C^0-iteration group of a function f such that $f(x) < x$, $x \in (0,1)$, then there exist an increasing homeomorphism $\alpha : [0,1) \to [0,\infty)$ and a decreasing homeomorphism $\beta : (0,1] \to [0,\infty)$ such that $\alpha(0) = 0$, $\beta(1) = 0$ and*

$$f^t(x) = \alpha^{-1}\big(s^t \alpha(x)\big), \quad x \in [0,1) \quad \text{and} \quad f^t(x) = \beta^{-1}\big(M^t \beta(x)\big), \quad x \in (0,1]. \tag{6}$$

Homeomorphisms α and β are determined uniquely up to a multiplicative constant. If $f(x) > x$, $x \in (0,1)$ the roles of α and β mutually change.

Define the following auxiliary family of one-parameter functions

$$\Gamma(c,x) := \frac{cx}{(1+(c^k-1)x^k)^{\frac{1}{k}}}, \quad x \in [0,1],\ c > 0.$$

Proposition 10. *Let $k > 0$ and f satisfy the hypothesis* (H). *If $\{f^t,\ t \in \mathbb{R}\}$ is a C^0-iteration group of f, then there exists homeomorphism $\psi : [0,1] \to [0,1]$ such that* (2) *holds, where $\{p_k^t,\ t \in \mathbb{R}\}$ is a group given by* (1). *Homeomorphism ψ is uniquely determined up to a constant. More precisely, if homeomorphisms ψ_1 and ψ_2 are of the same monotonicity and fulfill* (2), *then there exists a $c > 0$ such that $\psi_1 = \Gamma(c, \psi_2)$.*

Conversely, formula (2) *with an arbitrary homeomorphism ψ of* $[0,1]$ *defines a C^0-iteration group with two fixed points.*

Proof. Assume that $f(x) < x$ on $(0,1)$. Let $0 < s < 1$ and α be a homeomorphism determined in Lemma 9.
Put $\alpha(1) := \infty$ and $\psi := H_k^{-1} \circ \alpha$. By Lemmas 9 and 7 we have

$$f^t = \alpha^{-1} \circ S^t \circ \alpha = \alpha^{-1} \circ H_k \circ p_k^t \circ H_k^{-1} \circ \alpha = \psi^{-1} \circ p_k^t \circ \psi, \quad t \in \mathbb{R}.$$

To prove the uniqueness let us assume that ψ_1 and ψ_2 are homeomorphisms of $[0,1]$ such that

$$f^t = \psi_j^{-1} \circ p_k^t \circ \psi_j \quad \text{for } j \in \{1,2\} \text{ and } t \in \mathbb{R}.$$

Then

$$\frac{s^t \psi_1(x)}{[1+(s^{kt}-1)(\psi_1(x))^k]^{\frac{1}{k}}} = \psi_1 \circ \psi_2^{-1}\left(\frac{s^t \psi_2(x)}{[1+(s^{kt}-1)(\psi_2(x))^k]^{\frac{1}{k}}}\right).$$

After substitution $\gamma := \psi_1 \circ \psi_2^{-1}$ this equation can be written as

$$\frac{s^t \gamma(x)}{[1+(s^{kt}-1)(\gamma(x))^k]^{\frac{1}{k}}} = \gamma\left(\frac{s^t x}{(1+(s^{kt}-1)x^k)^{\frac{1}{k}}}\right)$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$. Putting in this equality an arbitrary $x_0 \in (0,1)$ and, after that, in turn, $d := \gamma(x_0)$, $y := s^t$ leads to the formula

$$\frac{yd}{(1+(y^k-1)d^k)^{\frac{1}{k}}} = \gamma\left(\frac{yx_0}{(1+(y^k-1)x_0^k)^{\frac{1}{k}}}\right) \quad \text{for } t > 0.$$

Let $z := \frac{yx_0}{(1+(y^k-1)x_0^k)^{\frac{1}{k}}}$. Consequently $z^k = \frac{y^k x_0^k}{1+(y^k-1)x_0^k}$ and, what follows, $y^k = \frac{z^k(1-x_0^k)}{x_0^k(1-z^k)}$. Using this substitution in the last equality we get

$$\gamma(z) = \left(\frac{\frac{z^k(1-x_0^k)}{x_0^k(1-z^k)} d^k}{1+(\frac{z^k(1-x_0^k)}{x_0^k(1-z^k)}-1)d^k}\right)^{\frac{1}{k}} = \left(\frac{z^k \frac{1-x_0^k}{x_0^k(1-d^k)} d^k}{1+(\frac{(1-x_0^k)d^k}{x_0^k(1-d^k)}-1)z^k}\right)^{\frac{1}{k}}, \quad z \in (0,1)$$

which with $c^k := \frac{(1-x_0^k)d^k}{x_0^k(1-d^k)}$ gives

$$\gamma(z) = \frac{cz}{(1+(c^k-1)z^k)^{\frac{1}{k}}}, \quad z \in (0,1).$$

Hence

$$\psi_1(x) = \frac{c\psi_2(x)}{[1+(c^k-1)(\psi_2(x))^k]^{\frac{1}{k}}} = \Gamma\big(c, \psi_2(x)\big).$$

Since γ is increasing ψ_1 and ψ_2 must be of the same monotonicity. $\square$

Remark 11. Let $f(x) \neq x$ on $(0,1)$ and $s \in (0,\infty) \setminus \{1\}$. The conjugating function ψ in formula (2) is increasing if and only if $(x - f(x))(1-s) \geqslant 0$ on $(0,1)$.

In particular case $k = 1$ we have $q^t = p_k^t$, $t \in \mathbb{R}$, hence we get

Corollary 12. *If $\{f^t,\ t\in\mathbb{R}\}$ is a C^0-iteration group of f satisfying* (H) *then such a group is conjugated to the group $\{q^t,\ t\in\mathbb{R}\}$. More precisely, there exists a homeomorphism $\phi:[0,1]\to[0,1]$ such that*

$$f^t(x)=\phi^{-1}\left(\frac{s^t\phi(x)}{1+(s^t-1)\phi(x)}\right),\quad x\in[0,1],\ t\in\mathbb{R}. \tag{7}$$

Corollary 13. *Let $\{f^t,\ t\in\mathbb{R}\}$ be a C^0-iteration group of f satisfying* (H). *The functions ψ and ϕ conjugating $\{f^t,\ t\in\mathbb{R}\}$ respectively with $\{p_k^t,\ t\in\mathbb{R}\}$ and $\{q^t,\ t\in\mathbb{R}\}$ satisfy the following functional equations*:

$$\text{(I)}\qquad \psi\big(f(x)\big)=\frac{s\psi(x)}{[1+(s^k-1)(\psi(x))^k]^{\frac{1}{k}}}$$

and

$$\text{(II)}\qquad \phi\big(f(x)\big)=\frac{s\phi(x)}{1+(s-1)\phi(x)}\quad \textit{for } x\in[0,1].$$

If, moreover, the function f is differentiable, $s\neq 1$ is positive and ψ is a diffeomorphism, then $s=f'(0)$ and $k=-\frac{\log M}{\log s}$, where $M:=f'(1)$.

Proof. Formula (I) we get taking $t=1$ in (2). Formula (II) is a simple form of (I) for $k=1$. To prove “the moreover” part observe that making use of (2) with $t=1$ and Remark 3 with $g=p_k^1$ we have $f'(0)=(p_k^1)'(0)=s$ and $M=f'(1)=(p_k^1)'(1)=\frac{1}{s^k}$. □

Corollary 14. *Let $f:[0,1]\to[0,1]$ be differentiable and $s\neq 0$. If Eq.* (II) *possesses a diffeomorphic solution on $[0,1]$ then $f'(0)=\frac{1}{f'(1)}$.*

Proof. According to Remark 3 we have $f'(0)=(q^1)'(0)=s$ and $f'(1)=(q^1)'(1)=\frac{1}{s}$. □

Let us note that if f is a given diffeomorphism satisfying (H) and ψ is a diffeomorphic solution of Eq. (I) which maps $[0,1]$ onto $[0,1]$, then formula (2) defines a C^1-iteration group of f. In further part of the paper we consider the problem of the uniqueness of these groups.

Proposition 15. *Let $\{f^t,\ t\in\mathbb{R}\}$ be a C^0-iteration group on $[0,1]$ of f satisfying* (H), *$f\leqslant id$, $0<s<1<M$ and $k=\frac{-\log M}{\log s}$. If $\psi:[0,1]\to[0,1]$ is a homeomorphism satisfying* (2), *then there exist homeomorphisms $\alpha:[0,1)\to[0,\infty)$ increasing and $\beta:(0,1]\to[0,\infty)$ decreasing satisfying* (6) *and such that*

$$\psi(x)=\frac{\alpha(x)}{(1+(\alpha(x))^k)^{\frac{1}{k}}},\quad x\in[0,1) \tag{8}$$

and

$$\psi(x)=\frac{1}{(1+d\beta(x))^{\frac{1}{k}}},\quad x\in(0,1], \tag{9}$$

where d is a positive constant.

Proof. Let a homeomorphism ψ satisfies (2). From Remark 11 it follows that ψ is increasing. Put $\alpha:=H_k\circ\psi$. Then $\psi=H_k^{-1}\circ\alpha$, whence we get (8) and

$$f^t=\alpha^{-1}\circ H_k\circ p_k^t\circ H_k^{-1}\circ\alpha$$

and, consequently, by Lemma 7, $f^t(x)=\alpha^{-1}(s^t\alpha(x))$, $x\in[0,1)$. Now let β be a homeomorphism determined in Lemma 9. By (6) we get

$$\alpha^{-1}\big(s^t\alpha(x)\big)=\beta^{-1}\big(M^t\beta(x)\big),\quad x\in(0,1),\ t\in\mathbb{R},$$

which with $\delta:=\beta\circ\alpha|_{(0,1)}^{-1}$ can be written as

$$\delta\big(s^tx\big)=M^t\delta(x).$$

Since $k = -\frac{\log M}{\log s}$, we get $M^t = \frac{1}{s^{kt}}$, and the last equation has the form

$$\big(\delta\big(s^t x\big)\big)^{-\frac{1}{k}} = s^t\big(\delta(x)\big)^{-\frac{1}{k}}.$$

Putting $\sigma := \delta^{-\frac{1}{k}}$ we get

$$\sigma\big(s^t x\big) = s^t \sigma(x), \quad x \in (0,1),\ t \in \mathbb{R}.$$

Since $\sigma(yx) = y\sigma(x)$, $x \in (0,1)$, $y > 0$, so there exists a constant a such that $\sigma(x) = ax$. Thus $\delta(x) = Ax^{-k}$, where $A = a^{-k}$. Consequently, $\alpha(x)^k = \frac{A}{\beta(x)}$ on $(0,1)$. Putting this to (8) gives (9) with $d := \frac{1}{A}$. □

Theorem 16. *If $\{f^t,\ t \in \mathbb{R}\}$ is a C^r-iteration group on $(0,1)$ without fixed points then there exists a C^r-diffeomorphism $\psi : (0,1) \to (0,1)$ such that the group is given by formula* (2).

Proof. (See also [2, Theorem 2.15].) From Lemma 9 it follows that $f^t(x) = \alpha^{-1}(s^t\alpha(x))$ in $(0,1)$ for $t \in \mathbb{R}$ with some homeomorphism $\alpha : (0,1) \to (0,\infty)$ and arbitrarily fixed $s \in (0,\infty) \setminus \{1\}$. Putting $\gamma(x) := \frac{\log \alpha(x)}{\log s}$ we obtain that $f^t(x) = \gamma^{-1}(t + \gamma(x))$. Blanton and Baker in paper [3] proved that if f^t which are expressed by the last formula are all of class C^r, then γ is of class C^r. Note that $\gamma'(x) \neq 0$ on $(0,1)$. In fact, we have $\gamma'(f^t(x))(f^t)'(x) = \gamma'(x)$ on $(0,1)$. If $\gamma'(x_0) = 0$ for an $x_0 \in (0,1)$ then $\gamma'(f^t(x_0)) = 0$ for $t \in \mathbb{R}$. Hence $\gamma' = 0$ since $h(t) := f^t(x_0)$ maps $\mathbb{R}$ onto $(0,1)$, but this is a contradiction. Thus γ is a C^r-diffeomorphism. Hence

$$\psi(x) := \frac{\alpha(x)}{(1+\alpha(x)^k)^{\frac{1}{k}}} = \frac{s^{\gamma(x)}}{(1+s^{k\gamma(x)})^{\frac{1}{k}}}$$

with arbitrarily fixed $k > 0$ is a C^r-diffeomorphism in $(0,1)$. It is easy to verify that $f^t = \psi^{-1} \circ p_k^t \circ \psi$ for $t \in \mathbb{R}$. □

4. C^1-iteration groups

In this section we give the proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $\{f^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$ be a C^1-iteration group such that $f^1 = f$. Differentiating of both sides of equality $f^u(f^v(x)) = f^{u+v}(x)$ at zero gives $(f^u)'(0)(f^v)'(0) = (f^{u+v})'(0)$, since $f^v(0) = 0$. Putting $A(t) := (f^t)'(0)$ we get

$$A(u)A(v) = A(u+v), \quad u, v \in \mathbb{R}$$

and $A(1) = f'(0) = s$. The mapping $t \mapsto (f^t)'(0)$ is measurable as the limit of continuous mappings $t \mapsto \frac{f^t(x_n)}{x_n}$, where $x_n \to 0+$. Hence $(f^t)'(0) = A(t) = s^t$ (see [1, p. 38]). Thus for every $t \in \mathbb{R}$

$$\lim_{x \to 0+} \frac{f^t(x)}{x} = s^t.$$

By Lundberg's theorem (see [9] and [6, Theorem 9.2]) applied to mapping $f|_{[0,1)}$ we know that

$$f^t(x) = \alpha^{-1}\big(s^t \alpha(x)\big), \quad \text{for } x \in [0,1),\ t \in \mathbb{R}, \tag{10}$$

where for a given $y_0 \in (0,1)$

$$\alpha(x) = u \lim_{n \to \infty} \frac{f^n(x)}{f^n(y_0)}, \quad x \in [0,1) \tag{11}$$

with arbitrarily chosen $u > 0$ since formula (10) holds independently of the choice of u. Obviously α satisfies the Schröder's equation

$$\alpha\big(f(x)\big) = s\alpha(x), \quad x \in [0,1) \tag{12}$$

and α maps $[0,1)$ onto $[0,\infty)$. Since $f'(x) = s + O(x^\delta)$, $x \to 0$, by Szekeres' theorem (see [10], [8, Theorem 3.5.1]), it is known that Eq. (12) has a C^1 solution γ such that $\gamma'(x) > 0$ for $x \in [0,1)$ and

$$\gamma(x) = \lim_{n \to \infty} \frac{f^n(x)}{s^n}, \quad x \in [0,1).$$

Hence, by (11), $\alpha(x) = u\frac{\gamma(x)}{\gamma(x_0)}$ for $x \in [0,1)$. Thus $\alpha \in \mathit{Diff}^1[0,1)$.

Let now ψ be a homeomorphism determined in Proposition 10, that is such that equality (2) holds. In a view of Proposition 15 we get

$$\psi(x) = \frac{\alpha_0(x)}{(1+\alpha_0(x)^k)^{\frac{1}{k}}}, \quad x \in [0,1),$$

where $\alpha_0 : [0,1) \to [0,\infty)$ is a homeomorphism such that $f^t(x) = \alpha_0^{-1}(s^t\alpha_0(x))$ in $[0,1)$. Take in (11) function α with $u = 1$. By (10) we have $\alpha^{-1}(s^t\alpha(x)) = \alpha_0^{-1}(s^t\alpha_0(x))$ for $x \in (0,1)$ and $t \in \mathbb{R}$. Putting $\nu := \alpha_0 \circ \alpha^{-1}$ we get $\nu(s^t x) = s^t\nu(x)$, $t \in \mathbb{R}$, $x \in (0,1)$, so $\nu(yx) = y\nu(x)$, $y > 0$. Hence $\nu(x) = cx$ for a $c > 0$. Consequently $\alpha_0 = c\alpha$ and we get

$$\psi(x) = \frac{c\alpha(x)}{(1+(c\alpha(x))^k)^{\frac{1}{k}}}, \quad x \in [0,1), \tag{13}$$

for a $c > 0$. Whence $\psi \in \mathit{Diff}^1[0,1)$ since $\alpha \in \mathit{Diff}^1[0,1)$.

Put $x_0 := \alpha^{-1}(\frac{1}{c})$. Then $c\alpha(x_0) = 1$ and

$$\lim_{n\to\infty}\frac{f^n(x)}{f^n(x_0)} = \lim_{n\to\infty}\frac{f^n(x)}{f^n(y_0)}\lim_{n\to\infty}\frac{f^n(y_0)}{f^n(x_0)} = \frac{\alpha(x)}{\alpha(x_0)} = c\alpha(x).$$

Hence formula (3) with given above x_0 is a simple consequence of Eq. (13).

We shall show that formula (2) holds for every ψ defined by (3) independently of the choice of x_0 in $(0,1)$. It follows, by (11), that for every $y \in (0,1)$ the sequence $(\frac{f^n(x)}{f^n(y)})_{n\in\mathbb{N}}$ is almost uniformly convergent in $[0,1)$. Hence there exists

$$\lim_{n\to\infty}\frac{f^n(x)}{[(f^n(y))^k + (f^n(x))^k]^{1/k}} =: \psi_y(x), \quad x \in [0,1]$$

and ψ_y is continuous in $[0,1)$. Taking into account the equality $\lim_{x\to 0+}\frac{f^t(x)}{x} = s^t$, commutativity of the composition $f^n \circ f^t$ and fact that $f^n(x) \to 0+$ as $n \to \infty$ it is easy to verify that

$$\psi_y\big(f^t(x)\big) = p_k^t\big(\psi_y(x)\big), \quad x \in [0,1],\ y \in (0,1),\ t \in \mathbb{R}. \tag{14}$$

Fix a $z_0 \in (0,1)$ and put $h(t) := f^t(z_0)$, $t \in \mathbb{R}$. In a view of Lemma 9, h maps $\mathbb{R}$ onto $(0,1)$ and is strictly decreasing. Moreover,

$$\psi_y\big(h(t)\big) = p_k^t\big(\psi_y(z_0)\big) \quad\text{and}\quad \psi_y(x) = p_k^{h^{-1}(x)}\big(\psi_y(z_0)\big), \quad x \in (0,1),$$

so ψ_y is strictly increasing and $\lim_{x\to 1-}\psi_y(x) = 1 = \psi_y(1)$. Hence ψ_y is a homeomorphism and, by (14),

$$f^t(x) = \psi_y^{-1}(p_k^t\big(\psi_y(x)\big), \quad x \in [0,1],\ y \in (0,1),\ t \in \mathbb{R}.$$

Thus the iteration group $\{f^t,\ t \in \mathbb{R}\}$ can be expressed by (2) with (3) for every $x_0 \in (0,1)$.

By the same argumentation used for function $f|_{(0,1]}$, in a view of Lundberg's theorem, we know that $f^t(x) = \beta^{-1}(M^t\beta(x))$ for $x \in (0,1]$, where

$$\beta\big(f(x)\big) = M\beta(x), \quad x \in (0,1] \tag{15}$$

and for a given $y_0 \in (0,1)$

$$\beta(x) = v\lim_{n\to\infty}\frac{f^{-n}(x)-1}{f^{-n}(y_0)-1}, \quad x \in (0,1] \tag{16}$$

for arbitrary $v > 0$. Formula (15) holds independently of the choice of $v > 0$. On the other hand, by Szekeres' theorem on the Schröder's equation, we know that (15) has C^1 solution σ such that $\sigma'(x) > 0$ for $x \in (0,1]$ and

$$\sigma(x) = \lim_{n\to\infty}\frac{f^{-n}(x)-1}{M^n}, \quad x \in (0,1].$$

Hence $\beta(x) = v\frac{\sigma(x)}{\sigma(y_0)}$ for $x \in (0,1]$ and, consequently, $\beta \in \mathit{Diff}^1(0,1]$. Further, similarly as in the previous case, by Proposition 15 formula (9) holds for a $d > 0$. Since $\beta(1) = 0$, $\psi \in \mathit{Diff}(0,1]$. Hence, combining the considered cases, we infer that $\psi \in \mathit{Diff}^1[0,1]$.

Similarly as previously, at first one proves that (16) and (9) imply (4) for an $x_0 \in (0,1)$ and then, again applying the method used above, we get our assertion for every $x_0 \in (0,1)$. $\square$

Remark 17. (See also [2, Theorem 2.15].) If a group $\{f^t,\ t \in \mathbb{R}\}$ is of class C^r, $r \geqslant 2$ without inner fixed points, then its conjugating function ψ with its canonical form is of class C^r.

The proof is the same as the previous one but, in addition, it is necessary to take into account Remark 11 and the fact that the Schröder's equations (12) and (15) have C^r solutions (see [6, p. 137]).

As a simple consequence of Theorem 1 we get the following

Corollary 18. *If groups $\{f^t,\ t \in \mathbb{R}\}$, $\{g^t,\ t \in \mathbb{R}\}$ are of class C^1 and at least one of elements of each of them satisfies condition* (R)*, then the groups are diffeomorphically conjugated.*

Based on the proof of Theorem 1 we obtain also the following result on the existence and the uniqueness of C^1 solutions of functional equation (I).

Theorem 19. *Let f satisfy* (R)*. Then for every $\nu > 0$ and $\omega > 0$ Eq.* (I) *with $k = -\frac{\log M}{\log s}$ has unique solutions $\psi_0 \in Diff^1[0,1)$ and $\psi_1 \in Diff^1(0,1]$ such that $(\psi_0)'(0) = \nu$ and $(\psi_1)'(1) = \omega$. Moreover, ψ_0 is given by formula* (3) *and ψ_1 is given by formula* (4)*.*

Proof. Let $\nu > 0$ and $\omega > 0$ be given. Define

$$\psi_0(x) := \frac{\nu\alpha(x)}{(1 + (\nu\alpha(x))^k)^{\frac{1}{k}}}, \quad x \in [0,1)$$

and

$$\psi_1(x) := \frac{1}{(1 + \omega\beta(x))^{\frac{1}{k}}}, \quad x \in (0,1],$$

where α is given by (11) with u such that $\alpha'(0) = 1$ and β is given by (16) with ν such that $\beta'(1) = -k$.

Previously we have shown that $\alpha \in Diff^1[0,1)$ and $\beta \in Diff^1(0,1]$ thus $\psi_0 \in Diff^1[0,1)$ and $\psi_1 \in Diff^1(0,1]$. It is easy to check that ψ_0 and ψ_1 satisfy (I) respectively on $[0,1)$ and $(0,1]$ and $\psi_0'(0) = \nu$ and $\psi_1'(1) = \omega$.

To prove the uniqueness assume that $\widetilde{\psi_0} \in Diff^1[0,1)$ satisfies (I) and $\widetilde{\psi_0'}(0) = \nu$. First note that (I) implies that either $\widetilde{\psi_0}(0) = 0$ or $\widetilde{\psi_0}(0) = 1$. Since $\widetilde{\psi_0}$ is strictly increasing we have $\widetilde{\psi_0}(x) > 0$ on $(0,1)$. Moreover, $\widetilde{\psi_0}$ satisfies the equation

$$\widetilde{\psi_0}\big(f(x)\big) = p\big(\widetilde{\psi_0}(x)\big), \quad x \in (0,1),$$

where

$$p(x) := \frac{sx}{[1 + (s^k - 1)x^k]^{1/k}}, \quad x \in [0,\infty).$$

Since p is bounded in $[0,\infty)$ the function $\widetilde{\psi_0}$ is also bounded. On the other hand it follows by (I) that $\lim_{x\to 1-} \widetilde{\psi_0}(x)$ equals either 0 or 1 or ∞. Since $\widetilde{\psi_0}$ is strictly increasing and bounded only the case $\lim_{x\to 1-} \widetilde{\psi_0}(x) = 1$ can occur. Then $\widetilde{\psi_0}(0) = 0$ and, consequently, $0 < \widetilde{\psi_0}(x) < 1$ for $x \in (0,1)$.

Now we may define

$$\alpha_0(x) := \frac{1}{\nu}\frac{\widetilde{\psi_0}(x)}{(1 - (\widetilde{\psi_0}(x))^k)^{\frac{1}{k}}}, \quad x \in [0,1).$$

It is easy to verify that α_0 satisfies (12), $\alpha_0 \in Diff^1[0,1)$ and $\alpha_0'(0) = 1$. By Szekeres' theorem we infer that $\alpha_0 = \alpha$ and this implies that $\widetilde{\psi_0} = \psi_0$.

Now assume that $\widetilde{\psi_1} \in Diff^1(0,1]$ satisfies (I) and $\widetilde{\psi_1'}(1) = \omega$. Similarly as in the previous case one shows that $\widetilde{\psi_1}(1) = 1$. Define

$$\beta_0(x) := \frac{1}{\omega}\frac{1 - (\widetilde{\psi_1}(x))^k}{(\widetilde{\psi_1}(x))^k}, \quad x \in (0,1].$$

It is easy to check that β_0 satisfies (15), $\beta_0 \in Diff^1(0,1]$ and $\beta_0'(1) = -k = \beta'(1)$. Similarly, by Szekeres' theorem we infer that $\beta_0 = \beta$ and, in consequence, $\widetilde{\psi_1} = \psi_1$. □

As a simple consequence of this theorem, Theorem 1 and Proposition 10 we get the following

Corollary 20. *If f satisfies* (R) *then f has at most one iteration group of class C^1 in* $[0,1]$*. More precisely, there exist exactly two groups $\{f_0^t,\ t \in \mathbb{R}\}$ and $\{f_1^t,\ t \in \mathbb{R}\}$ of f, the first one of class C^1 in* $[0,1)$ *and the second one of class C^1 in* $(0,1]$*. f is C^1-embeddable if and only if these groups coincide.*

Now we return to Theorem 2. To prove this theorem we use the following

Lemma 21. *(See [12].) Let a function $h \in Diff^1[0,1]$ be convex, satisfy* (H), $h'(0) < 1 < h'(1)$ *and* $x_0 \in (0,1)$. *Then there exists a C^1-iteration group $\{f^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$ such that*

$$f^1(x) = h(x) \quad \textit{for } x \in [0, x_0] \cup \bigl[h^{-2}(x_0), 1\bigr].$$

Proof of Theorem 2. Let

$$g(x) := s\int_0^x \left(1 - \frac{1}{\log t}\right) dt, \quad x \in [0,1),\ 0 < s < 1.$$

g is increasing and convex, $\lim_{x\to 1-} g(x) = \infty$. Observe also that $g'(0) = s < 1$. Hence there exists the only one fixed point $a \in (0,1)$ of g. Put $h(x) := \frac{1}{a} g(ax)$, $x \in [0,1]$. It is obvious that $h \in C^\infty[0,1]$ and $h'(0) = s$. Let $M := h'(1)$. Of course $M > 1$. Take an arbitrary $x_0 \in (0,1)$. Let $\{f^t : [0,1] \to [0,1],\ t \in \mathbb{R}\}$ be a C^1-iteration group such that

$$f^1\big|_{[0,x_0]\cup[h^{-2}(x_0),1]} = h|_{[0,x_0]\cup[h^{-2}(x_0),1]}.$$

Lemma 21 ensures the existence of this group. We show that the group $\{f^t,\ t \in \mathbb{R}\}$ is not diffeomorphically conjugated to any of groups $\{p_k^t, t \in \mathbb{R}\}$. Assume, in the contrary, that there exists a diffeomorphism $\psi : [0,1] \to [0,1]$ such that

$$f^t = \psi^{-1} \circ p_k^t \circ \psi, \quad t \in \mathbb{R}$$

for a $k > 0$. By Corollary 13 we get that $k = -\frac{\log M}{\log s}$. Let $h^t := f^t|_{[0,x_0]}$, $t \geqslant 0$. By Proposition 15 there exists a continuous, strictly increasing function $\alpha : [0,1) \to [0,\infty)$ satisfying (6) and (8). Since $\{h^t : [0,x_0] \to [0,x_0],\ t \in \mathbb{R}^+\}$ is a C^1-iteration semigroup such that $0 < (h^t)'(0) < 1$ for $t > 0$, by Lundberg's theorem,

$$h^t(x) = \gamma^{-1}\bigl(s^t \gamma(x)\bigr), \quad x \in [0, x_0],$$

where

$$\gamma(x) = \lim_{n\to\infty} \frac{h^n(x)}{h^n(x_0)}, \quad x \in [0, x_0].$$

On the other hand, by (10),

$$\gamma^{-1}\bigl(s^t \gamma(x)\bigr) = \alpha^{-1}\bigl(s^t \alpha(x)\bigr)$$

for all $x \in [0, x_0]$ and $t \geqslant 0$. Consequently, similarly as in the proof of Proposition 15, we infer that there exists an $r \in [0, x_0)$ such that $\alpha(x) = c\gamma(x)$ for $x \in [0, r]$ for some $c > 0$. On the other hand, by (8),

$$\alpha(x) = \frac{\psi(x)}{[1 - (\psi(x))^k]^{1/k}}, \quad x \in [0,1), \tag{17}$$

so α is of class C^1 in $[0,1)$, consequently γ is of class C^1 in $[0,r]$. In [7] and [6, p. 139] it is shown that the equation

$$\phi\bigl(h(x)\bigr) = s\phi(x), \quad x \in [0,1) \tag{18}$$

possesses infinitely many C^1 solutions and every C^1 solution ϕ of this equation satisfies the condition $\phi'(0) = 0$. Obviously γ satisfies (18), so $\gamma'(0) = 0$. On the other hand, by (17),

$$\frac{\gamma(x)}{x} = \frac{1}{c}\frac{\psi(x)}{x}\frac{1}{[1 - (\psi(x))^k]^{1/k}}, \quad x \in (0, r).$$

Letting $x \to 0$ we get

$$\gamma'(0) = \frac{1}{c}\psi'(0) \neq 0,$$

since $\psi(0) = 0$. This contradiction ends the proof. □

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